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HOMOLOGY AND HYPERGRAPH ACYCLICITY: A COMBINATORIAL INVARIANT FOR HYPERGRAPHS

BY A.D. PARKS
STRATEGIC SYSTEMS DEPARTMENT

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FOREWORD

This report describes certain theoretical results obtained from work performed jointly in the Strategic Systems (K) Department at the Naval Surface Warfare Center (NAVSWC) and in the Department of Mathematics at Mary Washington College under the auspices of an independent research grant entitled "A Category Theoretic Approach to Relational Database Schemes."

This report has been reviewed and approved by Ted Sims, Space Sciences Branch Head, and James L. Sloop, Space and Surface Systems Division Head.

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CONTENTS

INTRODUCTION

Because of their utility as relational database scheme models, certain classes of *acyclic* hypergraphs are important in a practical sense. These classes exhibit a nonreversible hierarchy, i.e., Berge-acyclicity $\Rightarrow$ γ-acyclicity $\Rightarrow$ β-acyclicity $\Rightarrow$ α-acyclicity **[1,2,4]**. From a purely graph-theoretic standpoint, these degrees of acyclicity can be characterized [3] by parallel degrees of chordality of simple graphs.

Here, the approach depends on a natural association of an abstract simplicial complex $\mathcal{S}_{\mathcal{H}}$ with a hypergraph $\mathcal{H}$, i.e., edges and their pairwise set intersections determine, respectively, abstract simplices and their pairwise common faces. We then choose a geometrical realization $K_{\mathcal{H}}$ of $\mathcal{S}_{\mathcal{H}}$ and the new invariant $\chi(\mathcal{H})$ is defined as the Euler number $\chi(K_{\mathcal{H}})$ **[7, p.124]**.

A new invariant for simple graphs is obtained by coupling the bijection $G \mapsto f(G)$ **[1,3]** (a graph G maps to the hypergraph $f(G)$ of its maximal cliques) with the hypergraph invariant. We calculate that χ has value one for hypergraphs that are connected α-acyclic by applying the Mayer-Vietoris sequence to show that the homology of the associated complex $K_{\mathcal{H}}$ is trivial — several corollaries immediately follow.

This construction is analogous to the well known construction [6] of the invariant $p = |V| - \sum_i^m (|E_i| - 1)$ for the class of Berge-acyclic ($\Rightarrow$ α-acyclic) hypergraphs. In the Berge case, a bipartite graph K_1 (a one-dimensional complex) is associated with a hypergraph and then the Euler number $\chi(K_1)$ is calculated. Here, a complex $K_{\mathcal{H}}$ (of dimension ≥ 1) is associated with a hypergraph $\mathcal{H}$ and then the Euler number $\chi(K_{\mathcal{H}})$ is calculated. We show (1) analogous to the Berge case, a hypergraph $\mathcal{H}$ is connected α-acyclic $\Rightarrow$ the complex $K_{\mathcal{H}}$ is acyclic $\Rightarrow$ $\chi(K_{\mathcal{H}}) = 1$; but (2) unlike the Berge case, these implications cannot (in general) be reversed. As a result of this, a new degree of hypergraph acyclicity, which we call h-acyclicity, is introduced and we have: $\mathcal{H}$ is h-acyclic $\Leftrightarrow$ $K_{\mathcal{H}}$ is homologically acyclic.

DEFINITIONS

A *hypergraph* $\mathcal{H}$ is a pair $(\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ is a finite set of vertices and $\mathcal{E}$ is a set of (hyper)edges which are nonempty subsets of $\mathcal{N}$. A hypergraph is *reduced* if no edge is a subset of another edge. It will be assumed here that all hypergraphs are reduced. A hypergraph is *α-acyclic* when it has the *running intersection property*, i.e., there is an ordering $(e_1, \ldots, e_n)$ of its edges such that for each i, $2 \leq i \leq n$, there is a $j_i < i$ such that $e_i \cap \left(\bigcup_{k<i} e_k\right) \subseteq e_{j_i}$. Further, a hypergraph has the *nonempty*

running intersection property when each of the $n-1$ intersections $e_i \cap \left(\bigcup_{k<i} e_k\right)$ is not empty. For hypergraphs, the nonempty running intersection property clearly implies both connectedness and α-acyclicity; and the converse follows by obtaining a separation when one of the $n-1$ intersections is empty. Indeed, if p is the number of connected pieces and q is the number of empty intersections $e_i \cap \left(\bigcup_{k<i} e_k\right)$, then $p = q + 1$.

Let G be a loopless, undirected simple graph with vertex set V. Then G is *chordal* if every cycle of length ≥ 4 has a *chord.* A *clique* of G is a complete subgraph of G. We let $\mathcal{M}(G)$ denote the family of subsets of V that induce maximal cliques of G. The correspondence $\mathcal{M} = \mathcal{M}(G)$ induces the bijection f which associates with each simple graph G the hypergraph $f(G) = (V, \mathcal{M})$ [1,3].

If S is a finite set, then the *closure of* S, $C\ell(S)$, is the family of nonempty subsets of S. The *closure of* $\mathcal{H}$, $C\ell(\mathcal{H})$, is the union of the closures of its edges, i.e., $C\ell(\mathcal{H}) = \bigcup_{E\in\mathcal{E}} C\ell(E)$. The number of sets of cardinality k in $C\ell(\mathcal{H})$, $C\ell(\mathcal{M}(G))$ is denoted h_k, g_k. Finally, the maximum k for which $h_k \neq 0, g_k \neq 0$ is denoted $\overline{\mathcal{H}}, \overline{\mathcal{M}}$.

Let $\{a_0, \ldots, a_k\}$ be a set of *geometrically independent points* in R^n. The *k-simplex* (or *simplex*) σ^k spanned by $\{a_0, \ldots, a_k\}$ is the set of points x in R^n for which there exist nonnegative real numbers $\lambda_0, \ldots, \lambda_k$ such that $x = \sum_{i=0}^{k} \lambda_i a_i$ and $\sum_{i=0}^{k} \lambda_i = 1$. In this case, $\{a_0, \ldots, a_k\}$ is called the *vertex set of* σ^k. A *face* of σ^k is any simplex spanned by a nonempty subset of $\{a_0, \ldots, a_k\}$. A *finite geometrical simplicial complex* (or *complex*) K is a finite set theoretic union of simplices such that: (1) every face of a simplex of K is in K; and (2) the nonempty intersection of any two simplices of K is a common face of each. Here we deal only with finite simplicial complexes. Thus, the dimension of K is the largest positive integer m such that K contains an m-simplex. The *vertex scheme* of K is the family of all vertex sets which span the simplices of K. If $\{L_i\}$ is a family of subcomplexes of K, then $\cup_i L_i$ and $\cap_i L_i$ (when not empty) are subcomplexes of K.

A *finite abstract simplicial complex* (or *abstract complex*) is a finite family $\mathcal{S}$ of finite nonempty sets such that if A is in $\mathcal{S}$, then so is every nonempty subset of A. Accordingly, the vertex scheme of a complex is an abstract complex; and when nonempty, a finite union of set closures and a finite intersection of set closures are abstract complexes.

Two abstract complexes $\mathcal{S}$ and $\mathcal{T}$ are isomorphic when a bijection ϕ from the vertex set of $\mathcal{S}$ onto the vertex set of $\mathcal{T}$ satisfies $\{a_0, \ldots, a_k\} \in \mathcal{S}$ if, and only if, $\{\phi(a_0), \ldots, \phi(a_k)\} \in \mathcal{T}$. Every abstract complex $\mathcal{S}$ is isomorphic to the vertex scheme of some geometrical simplicial complex K. The complex K is then called a *geometrical realization of* $\mathcal{S}$ and is uniquely determined (up to linear isomorphism). We denote an isomorphism between $\mathcal{S}$ and the vertex scheme of K by $\mathcal{S} \cong K$. An *edge* $e = \{u, v\} \in \mathcal{S}$ is a doubleton subset contained in $\mathcal{S}$. A distinct pair of vertices u, v of $\mathcal{S}$ are *path connected* if there is an alternating sequence $u\{u, x_1\}x_1\{x_1, x_2\} \ldots \{x_n, v\}v$ of vertices and edges of $\mathcal{S}$. The abstract complex $\mathcal{S}$ is *connected* when either $\mathcal{S}$ has only one vertex, or all pairs of its vertices are path

connected. If $\mathcal{S}$ is connected, so is any geometrical realization of $\mathcal{S}$.

To each simplicial complex K there corresponds a chain complex, i.e., for $p = 0, 1, 2, \ldots$ there are abelian groups $C_p(K)$ and homomorphisms $\partial_{p+1} : C_{p+1}(K) \to C_p(K)$. If K is finite and $\eta_p(K)$ is the number of p-simplices in K, then the *rank* of $C_p(K)$ is $\eta_p(K)$, i.e., $C_p(K)$ is isomorphic "$\cong$" to the direct sum "$\oplus$" of $\eta_p(K)$ copies of the additive group of integers Z. The p^{th} *homology group* of K, $H_p(K) = \ker \partial_p / \operatorname{im} \partial_{p+1}$, is finitely generated. The rank of $H_p(K)$ is the p^{th} *betti number* $b_p(K)$. If K is connected, then $H_0(K) \cong Z$. K is *homologically trivial* or *homologically acyclic* if $H_p(K) \cong 0$ whenever $p > 0$ and $H_0(K) \cong Z$. The complex of a simplex is homologically trivial.

PRELIMINARY LEMMAS

Several lemmas are required. The first three are well known and are repeated here for completeness. The fourth is due to DAtri, et al [3].

Lemma 1 **([7,p. 142]) (Mayer-Vietoris).** *Let L be a complex with subcomplexes K and L' such that $L = K \cup L'$. Then there is an exact sequence*

$$\cdots \to H_p(K \cap L') \to H_p(K) \oplus H_p(L') \to H_p(L) \to H_{p-1}(K \cap L') \to \cdots.$$

Lemma 2 **([8,p. 254]).** *Let A be an abelian group, F a free abelian group, and $\theta: A \to F$ be onto. Then $A \cong \ker \theta \oplus F$.*

Lemma 3 **([5,p. 242]) (Euler-Poincaré).** *If K is a complex of dimension m, then*

$$\sum_{p=0}^{m} (-1)^p \, \eta_p(K) = \sum_{p=0}^{m} (-1)^p \, b_p(K). \tag{1}$$

Lemma 4 **([3,p. 273]).** *A graph G is chordal if, and only if, $f(G)$ is α-acyclic.*

To obtain the new hypergraph invariant we start with the hypergraph $\mathcal{H} = (\mathcal{N}, \mathcal{E})$. Intuitively, we wish to make each edge $E \in \mathcal{E}$ into an abstract simplex — this amounts to considering the family $C\ell(E)$. And intuitively, we wish to make $\mathcal{H}$ into an abstract simplicial complex — this amounts to considering the family $C\ell(\mathcal{H})$. We then associate $C\ell(\mathcal{H})$ with one of its geometric realizations $K = K(C\ell(\mathcal{H}))$. The new hypergraph invariant can then be defined as the Euler-Poincaré invariant of this simplicial complex K, i.e., the Euler number or value of either the right or left side of (1). Similarly, the new invariant for graphs G can be defined as the Euler-Poincaré invariant of the simplicial complex $K(C\ell(f(G)))$. Consequently, the first step in constructing the hypergraph invariant involves the operator $C\ell$. The next lemma provides the key properties of $C\ell$. The proof is straightforward and is therefore omitted.

Lemma 5. *Let $\{S_i \mid i \in I\}$ be a collection of nonempty finite sets. Then the following statements are true.*

(1) $A \subset B \Leftrightarrow C\ell(A) \subset C\ell(B)$;
(2) $\cap_i C\ell(S_i) = C\ell(\cap_i S_i)$;
(3) $\cup_i C\ell(S_i) \subset C\ell(\cup_i S_i)$;
(4) $\emptyset \neq A \cap B \Leftrightarrow \emptyset \neq C\ell(A) \cap C\ell(B)$; and
(5) $\emptyset \neq A \Leftrightarrow \emptyset \neq C\ell(A)$.

Lemma 6. *The operator $C\ell$ preserves the (nonempty) running intersection property. That is, if $(S_1, \ldots, S_n)$ has the (nonempty) running intersection property, then the ordered set $(C\ell(S_1), \ldots, C\ell(S_n))$ has the (nonempty) running intersection property.*

PROOF: Write $S_i \cap (\cup_{k<i} S_k) = \cup_{k<i}(S_i \cap S_k)$. Then each addend on the right is a subset of S_{j_i}, and by (1) and (2) of Lemma 5 each addend $(C\ell(S_i) \cap C\ell(S_k))$ of the corresponding union is a subset of $C\ell(S_{j_i})$. Thus, the operator $C\ell$ preserves the running intersection property. To see that the "nonempty condition" is also preserved by $C\ell$, note that if one of the addends $S_i \cap S_k$ is not empty, then by (4) Lemma 5 the corresponding addend $C\ell(S_i) \cap C\ell(S_k)$ must also be nonempty. Thus, $C\ell$ also preserves each of the $n-1$ "nonempty conditions."∎

Lemma 7. *Let $(S_1, \ldots, S_n)$ have the running intersection property. If $S' = S_i \cap (\cup_{k<i} S_k)$, then*

$$C\ell(S') = C\ell(S_i) \cap (\cup_{k<i} C\ell(S_k)).$$

PROOF: Again, write $S' = S_i \cap (\cup_{k<i} S_k) = \cup_{k<i}(S_i \cap S_k)$. Then, for a nonempty set A, applications of Lemma 5 and the definition of $C\ell$ can justify the following statements and finish the proof:

$$\begin{aligned} A \in C\ell(S') \Leftrightarrow A \subset S' &\Leftrightarrow A \subset S_i \cap S_k \text{ for some } k < i \\ &\Leftrightarrow A \in C\ell(S_i \cap S_k) \text{ for some } k < i \\ &\Leftrightarrow A \in \cup_{k<i} C\ell(S_i \cap S_k) \\ &\Leftrightarrow A \in \cup_{k<i}(C\ell(S_i) \cap C\ell(S_k)) \\ &\Leftrightarrow A \in C\ell(S_i) \cap (\cup_{k<i} C\ell(S_k)). \blacksquare \end{aligned}$$

Lemma 8. *Let $\mathcal{H} = (\mathcal{N}, \mathcal{E})$ be a hypergraph. Then $S_{\mathcal{H}} = C\ell(\mathcal{H}) = \bigcup_{E \in \mathcal{E}} C\ell(E)$ is the **abstract simplicial complex associated** with $\mathcal{H}$. Let K be a geometrical realization of $S_{\mathcal{H}}$. Then $K_{\mathcal{H}} = K$ is called **a geometrical simplicial complex associated with** $\mathcal{H}$. If $(E_1, \ldots, E_n)$ is an ordering of $\mathcal{E}$ that satisfies the nonempty running intersection property,*

(1) *then let the simplex K_i of $K_{\mathcal{H}}$ correspond to the simplex $C\ell(E_i)$ of $S_{\mathcal{H}}$;*

(2) *then the subcomplex* L_i *of* $K_{\mathcal{H}}$ *corresponding to the subcomplex* $\cup_{k<i} C\ell(E_k)$ *of* $S_{\mathcal{H}}$ *is connected; and*

(3) *then the simplex* $K_i \cap L_i$ *of* $K_{\mathcal{H}}$ *necessarily corresponds to the simplex* $C\ell(E_i) \cap (\cup_{k<i} C\ell(E_k))$ *of* $S_{\mathcal{H}}$.

PROOF: Since $S_{\mathcal{H}} \cong K_{\mathcal{H}}$, we only need to show (1′) $C\ell(E_i)$ is a simplex of $S_{\mathcal{H}}$; (2′) $\cup_{k<i} C\ell(E_k)$ is a connected subcomplex of $S_{\mathcal{H}}$; and (3′) $C\ell(E_i) \cap (\cup_{k<i} C\ell(E_k))$ is nonempty and a simplex $S_{\mathcal{H}}$. But (1′) follows from $S_{\mathcal{H}} = \bigcup_{E \in \mathcal{E}} C\ell(E)$ and the definition of an abstract simplex; (2′) follows from an induction argument on $i \geq 2$ and the "nonempty condition"

$$\emptyset \neq C\ell(E_i) \cap \left(\cup_{k<i} C\ell(E_k)\right);$$

and (3′) follows from this "nonempty condition," Lemma 7, and the definition of an abstract simplex.∎

ACYCLIC HYPERGRAPHS

With the terminology developed above, we can prove:

Theorem 9. *If* $\mathcal{H}$ *is a connected* α*-acyclic hypergraph, then any associated geometrical simplicial complex* $K_{\mathcal{H}}$ *is homologically trivial.*

PROOF: Using the notation of Lemma 8, we let $i \in \{2, \ldots, n\}$ be fixed and consider the simplices $K_i \cap L_i$ and $K_i \cup L_i = L_{i+1}$. From Lemma 1 there is an exact sequence

$$\cdots \to H_p(K_i \cap L_i) \to H_p(K_i) \oplus H_p(L_i) \to H_p(L_{i+1}) \to H_{p-1}(K_i \cap L_i) \to \cdots.$$

Some of these groups are easily calculated. For example, since K_i and $K_i \cap L_i$ are simplices,

$$H_p(K_i) \cong H_p(K_i \cap L_i) \cong 0 \quad (p \geq 1) \tag{2}$$

and

$$H_0(K_i) \cong H_0(K_i \cap L_i) \cong Z. \tag{3}$$

Accordingly, substitution of (2) and (3) into the Mayer-Vietoris sequence induces two relevant exact sequences:

$$0 \longrightarrow H_p(L_i) \longrightarrow H_p(L_{i+1}) \longrightarrow 0 \quad (p \geq 2) \tag{4}$$

and

$$\cdots \to 0 \to H_1(L_i) \xrightarrow{\delta} H_1(L_{i+1}) \xrightarrow{\nu} Z \xrightarrow{\pi} Z \oplus H_0(L_i) \xrightarrow{\theta} H_0(L_{i+1}) \to 0 \to \cdots. \tag{5}$$

Since $i \geq 2$ is arbitrary, the exactness of (4) yields

$$H_p(L_2) \cong H_p(L_3) \cong \cdots \cong H_p(L_{n+1}) = H_p(K)\,. \quad (p \geq 2) \tag{6}$$

For $p \geq 2$ the first group $H_p(L_2)$ in (6) is 0 since L_2 is a simplex. Whence, (6) shows

$$H_p(K) = 0 \quad (p \geq 2). \tag{7}$$

To calculate the homology for K in dimension $p = 1$, we use the sequence (5): The exactness of (5) insures θ is surjective. Then Lemma 2, where $H_0(L_{i+1})$ free abelian and θ onto, and the exactness of (5) give

$$Z \oplus H_0(L_i) \cong \ker\theta \oplus H_0(L_{i+1}) = \operatorname{im}\pi \oplus H_0(L_{i+1}). \tag{8}$$

From Lemma 8 L_i, L_{i+1} are connected. Since $H_0(L_i) \cong Z \cong H_0(L_{i+1})$ we substitute Z into (8) and obtain

$$Z \oplus Z \cong \operatorname{im}\pi \oplus Z. \tag{9}$$

From (9) $\operatorname{im}\pi \cong Z$ and hence π is injective ($\ker\pi = 0$). So, the exactness of (5) at Z yields an exact sequence (4) for $p = 1$:

$$0 \longrightarrow H_1(L_i) \longrightarrow H_1(L_{i+1}) \longrightarrow 0 \quad (p = 1). \tag{10}$$

Therefore, since $i \geq 2$ is arbitrary, (10) shows that (6) also holds for $p = 1$. Repeating the argument above, for $p = 1$ the first group $H_p(L_2)$ in (6) is 0 since L_2 is a simplex. Ergo, (6) for $p = 1$ assigns

$$H_1(K) = 0. \tag{11}$$

Consequently, $L_{n+1} = K$ connected, (7), and (11) show K is homologically trivial.∎

Corollary 10. *If $\mathcal{H}$ is a connected α-acyclic hypergraph, then*

$$\sum_{k=1}^{\overline{\mathcal{H}}} (-1)^{k-1} h_k = 1. \tag{12}$$

PROOF: Because of the one-one correspondence between the $k-1$ dimensional simplices of $K = K_{\mathcal{H}}$ and the sets of size k in the family $S_{\mathcal{H}} = C\ell(\mathcal{H})$ we have $dim(K) = \overline{\mathcal{H}} - 1$ and $\eta_{k-1}(K) = h_k$. The validity of the equation (12) now follows from these observations, Theorem 1, and the appropriate substitutions in the Euler-Poincaré formula (1).∎

Corollary 11. *If G is a connected chordal graph, then*

$$\sum_{k=1}^{\overline{\mathcal{M}}} (-1)^{k-1} g_k = 1. \tag{13}$$

PROOF: First, we apply Lemma 4: Let the connected chordal graph G correspond to the connected α-acyclic hypergraph $f(G) = \mathcal{H}$. Observe that $g_k = h_k$ (and hence $\overline{\mathcal{M}} = \overline{\mathcal{H}}$). The validity of (13) is a consequence of these observations and appropriate substitutions in (12).∎

Corollary 12. *If $\mathcal{H}$ is an α-acyclic hypergraph, then $\chi(\mathcal{H})$ is the number p of components of $\mathcal{H}$.*

PROOF: Let $\{\mathcal{H}_t\}$ be the set of connected components of $\mathcal{H}$; let $\{K_t\}$ be a pairwise disjoint set of associated geometrical realizations, i.e., for each t, K_t is associated with $\mathcal{H}_t$. Let $K_{\mathcal{H}} = \bigvee_t K_t$ be the disjoint union of the K_t. Furthermore, let $(E_1, \dots, E_n)$ be an ordering of the edges of $\mathcal{H}$ that satisfies the running intersection property and, for each t, let V_t denote the vertex set of the component $\mathcal{H}_t$. Then the induced ordering of the nonempty sets in the list $(E_1 \cap V_t, \dots, E_n \cap V_t)$ is an ordering of the edges of $\mathcal{H}_t$ that satisfies the nonempty running intersection property. So, according to Theorem 9, $\chi(\mathcal{H}_t) = 1$ for each component $\mathcal{H}_t$. To finish the proof recall that the p^{th} betti number $b_p(K_{\mathcal{H}})$ of a disjoint union $K_{\mathcal{H}} = \bigvee_t K_t$ of complexes K_t is the sum $\sum_t b_p(K_t)$ of the betti numbers $b_p(K_t)$. ∎

EXAMPLES AND h-ACYCLIC HYPERGRAPHS

Analogous to the Berge case, we have (by Theorem 9):

$$\mathcal{H} \text{ connected } \alpha\text{-acyclic} \Rightarrow K_{\mathcal{H}} \text{ homologically acyclic } \Rightarrow \chi(K_{\mathcal{H}}) = 1. \tag{14}$$

Unlike the Berge case however, the two implications in (14) cannot (in general) be reversed. For sure, two simple examples suffice to show no two of the three statements in (14) are equivalent.

First, for $\mathcal{H}$ connected, we see

$$\mathcal{H} \text{ connected } \alpha\text{-acyclic} \not\Leftarrow K_{\mathcal{H}} \text{ homologically acyclic} \tag{15}$$

by considering the hypergraph $\mathcal{H} = (\mathcal{N}, \mathcal{E})$ with four vertices, $1, 2, 3, x$, and three edges $\{1,2,x\}$, $\{1,3,x\}$, $\{2,3,x\}$. Indeed, not one of the six possible orderings of $\mathcal{E}$ satisfies the running intersection property while $K_{\mathcal{H}}$ is isomorphic to a triangulation of a closed 2-disc.

Second, we show for $\mathcal{H}$ connected,

$$K_{\mathcal{H}} \text{ homologically acyclic} \not\Leftarrow \chi(K_{\mathcal{H}}) = 1. \tag{16}$$

To see that (16) holds, observe that for a given complex K, we can define $\mathcal{N}$ as the vertex set of K and we can define an edge set $\mathcal{E}$ by: $E \in \mathcal{E}$ when $E \subset \mathcal{N}$ and the vertices in E span a simplex of K. Then the hypergraph $\mathcal{H}_K = (\mathcal{N}, \mathcal{E})$ has K as an associated complex (if $\mathcal{E}$ contains only those subsets that span maximal simplices of K, then $\mathcal{H}$ would be reduced).

So for $\mathcal{H}$ connected (whence $K_{\mathcal{H}}$ connected), (16) is an instance of

$$K \text{ homologically acyclic} \not\Leftarrow \chi(K) = 1 \tag{17}$$

for a connected complex K. But (17) is well known: Identify one point of a 2-sphere with one point of a 1-sphere (circle) and let K be a triangulation of this quotient. Then

$$\chi(K) = \sum_p b_p(K) = b_0 - b_1 + b_2 = 1 - 1 + 1 = 1$$

but K is not homologically acyclic.

Since the first two simple statements of (14) are not equivalent, but the first implies the second, the concept of acyclicity in homology induces a new degree of acyclicity for hypergraphs.

More precisely, call a hypergraph *h-acyclic* when an associated complex has each of its connected components homologically acyclic. With this definition we have the following degrees of acyclicity for (not necessarily connected) hypergraphs:

$$\text{Berge-acyclic} \Rightarrow \gamma\text{-acyclic} \Rightarrow \beta\text{-acyclic} \Rightarrow \alpha\text{-acyclic} \Rightarrow h\text{-acyclic}. \qquad (18)$$

And each of the implications in (18) is (in general) not reversible.

REMARKS

In recent years, much research has been devoted to the study of θ-acyclic ($\theta = \alpha, \beta, \gamma$, Berge) relational database schemes. It has been shown that such schemes enjoy certain desirable properties, e.g., they have monotone join expressions. Because of their utility, the recognition of such schemes is an important design issue and elaborate recognition algorithms have been developed to assist in this capacity. As a fundamental application of our research, we note that since the Euler-Poincaré invariant is readily calculated for the simplicial complex associated with any relational database scheme, it may be used along with the contrapositive forms of Corollaries 10 and 12 to easily determine if a database scheme is not acyclic.

We define an h-acyclic database scheme as one having an h-acyclic hypergraph as its scheme model and observe that an arbitrary database scheme can be made h-acyclic by including an identical attribute in each relation. This follows since the resulting scheme has a cone as its associated simplicial complex and is thus homologically trivial.

A cover of a simplicial complex K is a family of subcomplexes $\mathcal{L} = \{ L_\alpha \mid \alpha \in A \}$ with $K = \cup_\alpha L_\alpha$. $\mathcal{L}$ is an acyclic cover if each L_α and each finite intersection $\cap_\alpha L_\alpha$ are homologically trivial. With this, we offer the following conjecture concerning conditions equivalent to h-acyclicity: **Conjecture.** *Let R be a database scheme. The following conditions on R are equivalent:*

(1) *R is an h-acyclic hypergraph.*
(2) *Graham's algorithm [1, p. 484] terminates with a set S such that $C\ell(S)$ is homologically trivial.*
(3) *$C\ell(R)$ has an acyclic cover with the running intersection property.*

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The natural associations from finite hypergraphs and simple graphs to finite simplicial complexes are used to produce new hypergraph and graph invariants. This correspondence shows that α-acyclic hypergraphs and chordal graphs map to homologically acyclic complexes. A new degree of hypergraph acyclicity (h-acyclicity where the h keys homology) is introduced. It is shown that α-acyclic $\Rightarrow$ h-acyclic, and an example is given to show that this implication is not reversible. Application of these results to database design is discussed and a conjecture characterizing h-acyclic database schemes is stated.

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